On a family of *A*-hypergeometric systems with exponential rank jumps

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Definition

- Gel'fand, Graev, Kapranov, Zelevinsky (1987-1989).
- An A-hypergeometric system is a system of linear partial differential equations associated with a pair (A, β), where

  $A = (a_1 \cdots a_n)$ is a full rank matrix, $a_i \in \mathbb{Z}^d$, $d \leq n$

  $\beta \in \mathbb{C}^d$ is a vector of complex parameters.

Definition

Toric ideal:

$$I_A = \langle \partial^{u_+} - \partial^{u_-} \; : \; u \in \mathbb{Z}^n, Au = 0 \rangle \subseteq \mathbb{C}[\partial]$$

where we denote $u = u_+ - u_-$, $u_+, u_- \in \mathbb{N}^n$ with disjoint support.

Example: $A = (2\ 3)$, $u = (3, -2)^t \in \ker A$, $I_A = \langle \partial_1^3 - \partial_2^2 \rangle$.

Definition

Euler operators: $E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n$ where $(a_{i1} \ \cdots \ a_{in})$ is the i-th row of A, $i = 1, \ldots, d$.

A-hypergeometric ideal:
$H_A(\beta) = DI_A + D\langle E_1 - \beta_1, \ldots, E_d - \beta_d\rangle \subseteq D$ where
$D = \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ is the Weyl Algebra.

Example: $A = (1\ 3\ 5)$, $I_A = \langle\partial_1^3 - \partial_2, \partial_1^5 - \partial_3\rangle$
$E - \beta = x_1\partial_1 + 3x_2\partial_2 + 5x_3\partial_3 - \beta$.

The rank of an A-hypergeometric system

- The rank of a system of linear P.D.E. is the dimension of the space of holomorphic solutions at nonsingular points.

- Assume $\mathbb{Z}A = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n = \mathbb{Z}^d$ and denote $\Delta_A = \text{convex-hull}(0, a_1, \ldots, a_n) \subseteq \mathbb{R}^d$.
  Normalized volume of A: $\text{vol}_{\mathbb{Z}^d}(A) = d!\,\text{vol}_{\mathbb{R}^d}(\Delta_A)$

- Gelfand-Kapranov-Zelevinsky, Adolphson, Saito-Sturmfels-Takayama: If $\beta \in \mathbb{C}^d$ is generic

$$\text{rank}(H_A(\beta)) = \text{vol}_{\mathbb{Z}^d}(A)$$

The rank of an A-hypergeometric system

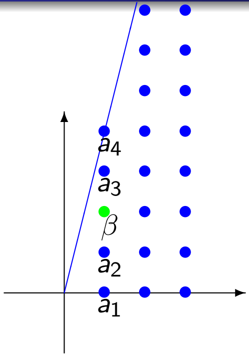
The first example of an A-hypergeometric system such that $\operatorname{rank}(H_A(\beta)) > \operatorname{vol}_{\mathbb{Z}^d}(A)$ (Sturmfels-Takayama, 1998):

$$A_{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In this example $\operatorname{rank}(H_{A_{(2)}}(\beta_{(2)})) = 5 > 4 = \operatorname{vol}_{\mathbb{Z}^2}(A_{(2)})$.

Reason: β is a hole in the semigroup $\mathbb{N}A_{(2)}$.

The rank of an A-hypergeometric system



Cattani-D'Andrea-Dickenstein (1999): If $d = 2$ then rank($H_{A_{(2)}}(\beta_{(2)})$) equals $\mathrm{vol}_{\mathbb{Z}^2}(A_{(2)}) + 1$ if β is a hole of $\mathbb{N}A_{(2)}$ or $\mathrm{vol}_{\mathbb{Z}^2}(A_{(2)})$ otherwise.

The rank of an A-hypergeometric system

- Saito-Sturmfels-Takayama (2000):
  $\mathrm{vol}_{\mathbb{Z}^d}(A) \leq \mathrm{rank}(H_A(\beta)) \leq 4^d \mathrm{vol}_{\mathbb{Z}^d}(A)$ for all β.

- Matusevich-Miller-Walther (2005): The map $\beta \in \mathbb{C}^d \mapsto \mathrm{rank}(H_A(\beta)) \in \mathbb{N}$ is upper-semi-continuous. The exceptional set
  $$\{\beta \in \mathbb{C}^d : \; \mathrm{rank}(H_A(\beta)) > \mathrm{vol}_{\mathbb{Z}^d}(A)\}$$
  is a finite union of affine subspaces.

The rank of an A-hypergeometric system

Matusevich-Walther (2007): A family of examples (A_d, β_d), $d \geq 2$, such that

- $\mathrm{vol}_{\mathbb{Z}^d}(A_{(d)}) = d + 2.$
- $\mathrm{rank}(H_A(\beta)) = 2d + 1$

Thus $\mathrm{rank}(H_A(\beta)) - \mathrm{vol}_{\mathbb{Z}^d}(A) = d - 1$ and
$\mathrm{rank}(H_{A_{(d)}}(\beta_{(d)}))/\,\mathrm{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = 2 - (3/(d+2)) < 2$
Case $d = 3$:

$$A_{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \text{ and } \beta_{(3)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

In this case, $\beta_{(3)}$ is again a hole in the semigroup $\mathbb{N}A_{(3)}$.

The rank of an A-hypergeometric system

- Saito (2002), Okuyama (2006): formula for $\mathrm{rank}(H_A(\beta))$ under some assumptions.
- Berkesch (2011): General formula for $\mathrm{rank}(H_A(\beta))$.
- However, $\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A) < 2$ in the examples in the literature.
- We construct examples where $\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A)$ is exponential in d.

Construction of the examples

Recall that the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1 \times n_1}, A_2 \in \mathbb{Z}^{d_2 \times n_2}$ is the following $(d_1 + d_2) \times (n_1 + n_2)$ matrix:

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0_{d_1 \times n_2} \\ 0_{d_2 \times n_1} & A_2 \end{pmatrix}$$

where $0_{d \times n}$ denotes the $d \times n$ zero matrix.

Lemma

If A is the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1 \times n_1}, A_2 \in \mathbb{Z}^{d_2 \times n_2}$ then $\mathrm{vol}_{\mathbb{Z}A}(A) = \mathrm{vol}_{\mathbb{Z}A_1}(A_1) \cdot \mathrm{vol}_{\mathbb{Z}A_2}(A_2)$.

Construction of the examples

Lemma

The A-hypergeometric system with $A = A_1 \oplus A_2$ and parameter $\beta = (\beta_1, \beta_2)$ decomposes as two hypergeometric systems for the same unknown (but disjoint sets of x_i, ∂_i-variables in the operators) corresponding to the pairs (A_1, β_1) and (A_2, β_2) respectively.

Corollary

For $A = A_1 \oplus A_2$ and parameter $\beta = (\beta_{(1)}, \beta_{(2)})$ we have $\mathrm{rank}(H_A(\beta)) = \mathrm{rank}(H_{A_1}(\beta_{(1)})) \cdot \mathrm{rank}(H_{A_2}(\beta_{(2)}))$.

Construction of the examples

For $A = A_1 \oplus A_2$ and parameter $\beta = (\beta_{(1)}, \beta_{(2)})$ we have

$$\frac{\operatorname{rank}(H_A(\beta))}{\operatorname{vol}_{\mathbb{Z}^d}(A)} = \frac{\operatorname{rank}(H_{A_1}(\beta_{(1)}))}{\operatorname{vol}_{\mathbb{Z}A_1}(A_1)} \cdot \frac{\operatorname{rank}(H_{A_2}(\beta_{(2)}))}{\operatorname{vol}_{\mathbb{Z}A_2}(A_2)}$$

1. Take $A_1 \in \mathbb{Z}^{d_1 \times n_1}$ and $\beta_1 \in \mathbb{C}^{d_1}$ with $\dfrac{\operatorname{rank}(H_{A_1}(\beta_{(1)}))}{\operatorname{vol}_{\mathbb{Z}^{d_1}}(A_1)} > 1$.
2. Consider $A = \oplus_1^r A_1$ and $\beta = (\beta_1, \ldots, \beta_1)$
3. $\operatorname{rank}(H_A(\beta))/\operatorname{vol}_{\mathbb{Z}^d}(A)$ is exponential on d.

Construction of the examples

Theorem

There is a family of hypergeometric systems $H_A(\beta)$ *such that* $\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A)$ *grows like* $(\sqrt[3]{7/5})^d$ *where* $d = \mathrm{rank}(A) \geq 2$.

All the examples of this family of hypergeometric systems decompose in examples of smaller hypergeometric systems.

We can construct a different family with rank/volume exponential on d.

Let us see how for $d = 4$.

Modifying the examples

Berkesch (2011)

$\mathrm{rank}(H_A(\beta))$ depends only on $\beta \in \mathbb{C}^d$, $\mathbb{N}A$ and Δ_A.

Thus, given a matrix A we can add a column $b \in \mathbb{N}A \cap \Delta_A$.
Case $d = 4$ (previous family):

$$A = A_{(2)} \oplus A_{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

Modifying the examples

Case $d = 4$ (previous family): $A = A_{(2)} \oplus A_{(2)}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}.$$

Here $\sqrt[d]{\operatorname{rank}(H_A(\beta))/\operatorname{vol}_{\mathbb{Z}^d}(A)} = \sqrt{5/4}$

Modifying the examples

Case $d = 4$ (previous family):

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Here $\sqrt[d]{\operatorname{rank}(H_A(\beta))/\operatorname{vol}_{\mathbb{Z}^d}(A)} = \sqrt{5/4}$

Modifying the examples

Case $d = 4$ (second family):

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Here $\sqrt[d]{\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A)} = \sqrt{9/8}$

Modifying the examples

Case $d = 4$ (last family):

$$A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 4 & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

Here $d = 4$, $\sqrt[4]{\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A)} = \sqrt{9/8}$ and A is not a direct sum of smaller matrices.
Similarly for all $d \geq 4$, $\mathrm{rank}(H_A(\beta))/\mathrm{vol}_{\mathbb{Z}^d}(A) \geq (\sqrt{9/8})^d$.

Open question

While the known upper bound for $\operatorname{rank}(H_A(\beta)/\operatorname{vol}_{\mathbb{Z}^d}(A)$ is 4^d (Saito-Sturmfels-Takayama), for the worst (or best) family we get that rank/volume grows like $(\sqrt[3]{7/5})^d$.

It is still an open problem to find a sharp upper bound for $\operatorname{rank}(H_A(\beta)/\operatorname{vol}_{\mathbb{Z}^d}(A)$.

Based on the paper

Exponential growth of rank jumps for A-hypergeometric systems.To appear in Revista Matemática Iberoamericana. (See also arXiv:1201.5090v1 [math.AG])

Thank you!